13 MAINTENANCE AND REPLACEMENT MODELS UNDER A FUZZY FRAMEWORK

Augustine O. Esogbue
Warren E. Hearnes II

Appears in *Handbook of Fuzzy Sets, Volume 5, Methodology 3: Operations Research and Statistics*, R. Slowinski (editor).

Abstract:

Uncertainty is present in virtually all replacement decisions due to unknown future events, such as revenue streams, maintenance costs, and inflation. Fuzzy sets provide a mathematical framework for explicitly incorporating imprecision into the decision-making model, especially when the system involves human subjectivity. This chapter illustrates the use of fuzzy sets and possibility theory to explicitly model uncertainty in replacement decisions via fuzzy variables and fuzzy numbers. In particular, a fuzzy set approach to *economic life of an asset* calculation as well as a finite horizon single asset replacement problem with multiple challengers is discussed. Because the use of triangular fuzzy numbers provides a compromise between computational efficiency and realistic modeling of the uncertainty, this discussion emphasizes fuzzy numbers. The algorithms

used to determine the optimal replacement policy incorporate fuzzy arithmetic, dynamic programming with fuzzy rewards, the vertex method, and various ranking methods for fuzzy numbers. A brief history of replacement analysis, current conventional techniques, the basic concepts of fuzzy sets and possibility theory, and the advantages of the fuzzy generalization are also discussed.

13.1 ECONOMIC DECISION ANALYSIS

Economic decision analysis is a useful tool, offering individuals and organizations the techniques to model economic decision-making problems, such as maintenance and replacement decisions, and determine an optimal decision. However, the accuracy of the model determines the validity of the conclusion. In many cases, the assumption of certainty in many models is made not so much for validity but the need to obtain simpler and more readily solvable formulations. Essentially, the tradeoff is between an *inaccurate but solvable* model and a *more accurate but potentially unsolvable* one. In most real-world systems, however, there are elements of uncertainty in the process or its parameters, which may lack precise definition or precise measurement, especially when the system involves human subjectivity.

When developing a model of a system with uncertainty, the decision-maker can either *ignore* the uncertainty, *implicitly acknowledge* it, or *explicitly model* it. Ignoring the uncertainty usually results in a deterministic model of the process with precise values for all parameters. Implicitly acknowledging the uncertainty may still result in a deterministic model in which sensitivity analysis or discount factors can be used to get an idea of how this uncertainty affects the outcome. Lastly, the decision-maker can explicitly model the uncertainty using specific paradigms such as interval analysis, possibility theory, probability theory, or evidence theory [Behrens and Choobineh, 1989].

The proper paradigm depends on the nature of the uncertainty. When the probabilities are specified for the outcomes, then the theory of Von Neumann and Morgenstern [Von Neumann and Morgenstern, 1944] provides the tools necessary to determine the optimal decision. However, in many cases these probabilities are neither defined nor directly attainable. Under these circumstances, other theories are needed. The most common choice is the use of subjective probability distributions and the theory of choice due to Savage [Savage, 1954]. However, considerable debate on the use of subjective probabilities exists and is well documented in the literature [Bezdek, 1994; Dubois and Prade, 1994; Klir, 1994; Kosko, 1994; Laviolette and Seaman, 1994]. From a psychological standpoint, the methods used to elicit these subjective probabilities and the validity of the subjective probabilities themselves have been the focus of research led by Tversky and Kahneman [Tversky and Kahneman, 1971; Tversky and Kahneman, 1981; Tversky and Kahneman, 1974]. Their studies show that

the heuristics employed to assess probabilities and predict values can sometimes lead to "severe and systematic errors" [Tversky and Kahneman, 1974].

Because humans do not think naturally in probabilistic terms, they tend to find the notions of fuzzy sets and their linguistic based approaches more user-friendly and appealing. We may view fuzzy set theory as a generalization of classical set theory since it provides us with a mathematical tool for describing sets that have no sharp transition from membership to nonmembership. Membership in a fuzzy set is defined by a generalized version of the classical indicator function called a membership function. Fuzzy sets allow the definition of vague or imprecise concepts such as "*approximately* 1000". This theory has been developed and successfully applied to numerous areas such as control, decision-making, engineering, and medicine. Its application to economic analysis is natural due to the uncertainty inherent in many financial and investment decisions. As noted earlier, it provides a precise mathematical language to model uncertainty due to vagueness and imprecision in events or statements describing a system. More information on fuzzy set theory, particularly fundamental concepts such as fuzzy numbers, which are invoked in our presentation, can be found in [Zadeh, 1978; Zimmerman, 1991].

13.2 REPLACEMENT ANALYSIS

One of the most practical and topical areas of engineering economics is replacement analysis. Mathematical models and analysis methods are used to determine the sequence of replacement decisions that provides a required service for a specified time horizon in an optimal manner. Maintenance and replacement decisions are assumed to occur on a periodic basis. The decision-maker chooses from various options, such as to keep, overhaul, or perform preventive maintenance on the existing asset or replace it with a new/used asset. Any sequence of decisions is called a *replacement policy* and any sequence that optimizes some performance measure such as net present value or annual equivalent cost is an *optimal replacement policy.*

In replacement analysis the *economic life of an asset* determines the replacement cycle that gives the minimum annual equivalent cost (MAEC) of operating a single asset over an infinite horizon [Sharp-Bette and Park, 1990]. Dynamic programming (DP), with discounting to put more emphasis on short-term income, is an acknowledged tool for the determination of the optimal replacement policy for more general replacement models [Gluss, 1972]. Using the DP approach some of the restrictive assumptions of the economic life method can be relaxed and still produce a computationally feasible solution algorithm. Early pioneers in the use of dynamic programming for finite horizon equipment replacement problems were Bellman and Wagner. Bellman [Bellman, 1955; Bell-

man and Dreyfus, 1962] was the first to formulate the replacement problem as a dynamic program. Optimal replacement policies were proposed first for the case with no technological change and then assuming some type of technological improvement. Bellman formulated a discounted DP version of the *economic life of an asset* model and determined analytically the optimal age T to replace the asset. In the more challenging technological improvement version, the revenue, the upkeep costs, and the replacement costs are assumed to be functions of the date the asset is installed as well as its age with respect to installation. Wagner [Wagner, 1975] formulated the replacement problem as a network and solved for the shortest path, which corresponded to the minimum outlay. Terborgh [Terborgh, 1949] included linear obsolescence in his formulation while Alchian [Alchian, 1952] allowed operating revenues and operating costs to increase linearly with time. Oakford [Oakford, 1970] modeled increasing revenues and data. Dreyfus [Dreyfus and Law, 1977] modeled technological change in revenue, maintenance, and replacement costs using bounded exponential functions. The Dynamic Replacement Economy Decision Model (DREDM) developed in [Oakford et al., 1984] is a generalization of Wagner's dynamic programming model that allows for multiple challengers and time-varying parameters.

Replacement models of great interest and relevance to this research are those that model uncertainty. Dreyfus and Law [Dreyfus and Law, 1977] treat the replacement problem where determinism yields to stochasticity. Their model includes the probability of a catastrophic failure in the asset being used as well as an uncertain net operating cost that is modeled by another probability distribution. The DP algorithm determines the minimum expected cost for the process. The Stochastic Replacement Economy Decision Model (SRE) presented by Lohmann [Lohmann, 1986] is a stochastic generalization of the DREDM. The assumption that the cash flows and relationships are known with certainty was relaxed and the component cash flows are modeled as triangular probability distributions based on the decision-maker's subjective judgment. The solution for this model is generated through Monte-Carlo simulation, which determines the probability that each asset is the optimal choice at time 0 as well as the probability distribution of the optimal net present value of the policy.

A tacit assumption implicit in the foregoing models is that uncertainty in the replacement decision can be fully modeled either deterministically or stochastically. This is not, however, always the case. Limiting replacement models to these two approaches either ignores the uncertainty or assumes that all uncertainty is probabilistic in nature and that the probabilistic information is fully known. Categorically classifying all uncertainty as randomness may not be reasonable or adequate.

In recent years the debate concerning the use of nonprobabilistic uncertainty, and specifically fuzzy sets, has surfaced in the area of economic analysis

[Behrens and Choobineh, 1989; Buckley, 1987; Buckley, 1992; Chiu and Park, 1994; Choobineh and Behrens, 1992; Gupta, 1993; Hearnes, 1995; Ward, 1985]. The replacement decisions made at each time period are based not only on the current cash flows but also on projected future cash flows of all possible assets [Lohmann, 1986]. Therefore, uncertainty in these cash flows can have a pronounced effect on the optimal replacement policy. A fuzzy set theoretic approach, as described in the sequel, may lead to more informed replacement decisions when the assumptions for a probabilistic approach are not met. For the deterministic case, a fuzzy set theoretic approach using fuzzy numbers is equivalent to multivariable sensitivity analysis and immediately provides both the deterministic optimal value and the possible range due to uncertainty.

13.3 FUZZY CONCEPTS IN CASH FLOW ANALYSIS

Cash flows, the basic variable in replacement decisions, are used by managers and financial analysts to measure the streams of money going into and flowing out of a particular organization's operation [Sharp-Bette and Park, 1990]. Traditionally, cash flows are treated as either deterministic or stochastic. However, as shown in simulation studies [Oakford et al., 1981], uncertain information in estimating these cash flows can limit the value of the analysis. Errors in deterministic cash flow estimations can skew the results of the analysis. Similarly, the use of subjective probability distributions is of concern since they generally cannot be verified, and the required historical information for generating frequency-based probability distributions is not always available.

The fundamental types of uncertainty, *nonspecificity*, *fuzziness*, and *strife*, are examined by Klir and Yuan [Klir and Yuan, 1995]. Uncertainty occurs in replacement and maintenance decisions in various ways. Of particular interest are nonspecificity and fuzziness which may factor into the estimates of the aggregate cash flows, the purchase prices/salvage values, the minimum attractive rate of return (MARR), or the physical lifetimes of the assets. This is especially true when these variables are based on the estimates provided by experts via the use of such natural language statements as "*approximately* \$1000". Through the use of fuzzy variables one can represent this vagueness and imprecision. In this presentation, however, these vague quantities will be represented using triangular fuzzy numbers (TFNs).

Several basic concepts of fuzzy sets applicable to the foregoing forms of uncertainty are *fuzzy sets*, α-*level sets*, *convexity*, and *triangular fuzzy numbers*. It is assumed that the reader is familiar with these concepts. If not, the reader is referred to [Zadeh, 1978; Zimmerman, 1991] for a detailed review. Fuzzy numbers are fuzzy sets defined on the set of real numbers, generally used to represent vague expressions, such as"*about* 20" or "*approximately* 1000", used

often in the description of uncertain economic decision systems. A special type of fuzzy number is the L-R fuzzy number introduced by Dubois and Prade [Dubois and Prade, 1978]. These types of fuzzy numbers simplify the arithmetic operations considerably. Triangular fuzzy numbers are a special type of L-R fuzzy number that are used in the models developed in this research. A *triangular fuzzy number* (TFN) is a fuzzy number $M_T = [l, m, r]$ defined by the membership function

$$\mu_{M_T}(x) = \begin{cases} (x-l)/(m-l), & \text{for } x \in (l, m] \\ (r-x)/(r-m), & \text{for } x \in (m, r) \\ 0, & \text{otherwise} \end{cases} \tag{13.1}$$

where m is the modal value and l and r are the left and right limits of the support for the fuzzy number, respectively.

13.3.1 Nonprobabilistic Methods in Cash Flow Analysis

Research into fuzzy versions of cash flows began with Ward [Ward, 1985], defining them as trapezoidal fuzzy numbers and solving a fuzzy present worth problem. Buckley [Buckley, 1987] used fuzzy numbers to develop fuzzy net present value (NPV) and fuzzy net future value (NFV) with fuzzy interest rates over a period of n years where n may also be fuzzy set. Buckley developed fuzzy equivalents to continuous interest payments, the effective rate of interest, and regular annuities as well. Restricting the fuzzy cash flows to positive fuzzy numbers allows the multiplication operation to be distributive over addition. The fuzzy number of time periods produces nonlinearities that make computations more complex. Li Calzi [Li Calzi, 1990] provided an axiomatic development for the fuzzy extension of financial mathematics with a desire to maintain consistency in the calculations. He examined two classes of fuzzy quantities, compact fuzzy intervals and invertible fuzzy intervals, and proved general theorems regarding consistency.

Two of the most recent and practical applications of nonprobabilistic uncertainty to economic analysis are given in [Chiu and Park, 1994; Choobineh and Behrens, 1992]. Choobineh and Behrens [Choobineh and Behrens, 1992] call attention to the use of intervals and possibility theory in economic analysis. The weak distributivity of interval arithmetic is noted, but a technique called the vertex method [Dong and Shah, 1987] is utilized to bypass this problem in interval and fuzzy arithmetic. Their approach to modeling cash flows as fuzzy intervals is similar to Ward's. Chiu and Park [Chiu and Park, 1994] use fuzzy numbers in cash flow analysis and provide a good survey of the major methods for ranking mutually exclusive fuzzy projects. The cash flows are modeled as triangular fuzzy numbers and the linear approximation to the product of two

triangular fuzzy numbers is investigated. The present worth of a fuzzy project is also examined. Their resultant formulation of a fuzzy present worth is

$$PW = \sum_{t=0}^{n} \left\{ (P_t) \left(\prod_{s=0}^{t} (1 + R_s) \right)^{-1} \right\} \tag{13.2}$$

where P_t is a positive or negative TFN representing the cash flow at the end of time t, n is the number of evaluation periods, and R_s is the nonnegative TFN representing the discount rate at the end of time s. Extending these ideas to replacement analysis, Hearnes [Hearnes, 1995] formulated fuzzy versions of the economic life of an asset model and the finite single asset replacement problem. This work is used as a point of departure for the discussions that follow.

13.3.2 Fuzzy Arithmetic and Interval Analysis

Fuzzy numbers represent vague notions of precise quantities. It is essential to be able to perform algebraic operations on them. The arithmetic operations of addition, subtraction, multiplication, and division developed by Dubois and Prade [Dubois and Prade, 1978] and particularly in [Dubois and Prade, 1987] are particularly useful when modeling and analyzing cash flows. The algebraic operations for TFNs are specifically reviewed. The choice of TFNs is in part due to their simplified algebraic operations. However, the set of TFNs is not closed under the operations of multiplication and division. The effect of using a linear (TFN) approximation, which is studied thoroughly in [Kaufmann and Gupta, 1988], is not significant for multiplication, inverse, and division operations. The TFN approximation to the multiplication operation is used in the following models for computational simplicity. The approximation error was also empirically determined to be not significant for the example problems in the sequel.

Fuzzy numbers are a family of nested intervals [Choobineh and Behrens, 1992] which correspond to levels of "confidence" by the decision-maker and therefore are closely related to interval analysis. However, like interval arithmetic, the multiplication operation for fuzzy numbers is only weakly distributive over addition. This presents a problem when modeling with fuzzy numbers since the outcome can depend on the form of the equation used. There are some special cases, however, where the multiplication operation is distributive over addition. If M_1, M_2, M_3 are fuzzy numbers, the multiplication operation is distributive over addition (i.e., $M_1 \cdot (M_2 + M_3) = M_1 \cdot M_2 + M_1 \cdot M_3$) when [Dubois and Prade, 1987]:

1. M_1 is a real number—i.e., scalar multiplication is distributive over addition.

2. M_2 and M_3 are both positive or both negative.

3. $M_2 = -M_2$ and $M_3 = -M_3$.

When conditions (1) – (3) are not met, a procedure called the *vertex method* [Dong and Shah, 1987] preserves the distributivity of multiplication over addition. The price, however, is an exponential increase in the number of computations.

13.3.3 Some Sources of Uncertainty in Replacement Analysis

Several aspects of replacement analysis contain imprecision and vagueness that warrant further discussion. We postulate that some of these variables, such as the physical lifetime of an asset, aggregate cash flow estimates, MARR, and inflation significantly impact the optimal replacement policy.

Physical Lifetime of an Asset. The physical lifetime of some assets may not be known with certainty, yet this is tacitly assumed and treated as deterministic in many models. In some situations, if enough information is known, then it is appropriate to treat it probabilistically. However, in cases where the asset is a new technology or a new model, this information is not generally available and such an approach must be considered suspect. In this case, it is instructive to treat the uncertainty in this variable through the use of fuzzy DP models but particularly those that allow stochastic or fuzzy termination times [Buckley, 1987; Kacprzyk, 1978]. For example, for an older asset the historical information about failures may be known and a probability distribution for failure can be derived. However, for an asset with new technology estimate of physical lifetime may be a fuzzy set such as "*about* 5 years" or "*more or less* 10 years". In each of these cases, the decision space has an uncertain boundary that affects the overall decision policy. Stochastic and fuzzy DP [Esogbue and Kacprzyk, 1996] provide methods for dealing with this type of uncertainty. Similarly, there may be uncertainty in the actual horizon N of the project which may be either stochastic or fuzzy. For example, the project duration of asset life may be determined if the state of the asset reaches some imprecisely definable point.

Aggregate Cash Flows. Another source of uncertainty in replacement and maintenance decisions is the estimation of the aggregate cash flow for each time period. In previous models, aggregate cash flows were treated as either deterministic or stochastic variables, but errors in these can lead to skewed analysis [Oakford et al., 1981]. Subjective probability distributions generally cannot be verified, while the required historical information for generating frequency-

based probability distributions is not generally available. In these cases, it may be more appropriate to define the aggregate cash flows as possibility distributions based on a decision-maker's opinion or expert judgment.

Minimum Attractive Rate of Return. The minimum attractive rate of return (MARR), which is usually used for project evaluation and comparison, is also another variable which may realistically possess forms of uncertainty [Sharp-Bette and Park, 1990]. However, in classical approaches this either is not addressed or is erroneously assumed to be well known or deterministic. The selection of the proper MARR plays an important role in the outcome of the maintenance and replacement decisions. There are a number of ways to determine a corporation's MARR, such as the use of the Delphi method involving its directors or some chosen mathematical formula. However, the MARR can be investment or management dependent. Because of the uncertainties characteristic of investment and management decision processes, it is inevitable that any MARR thus determined is imprecise or fraught with uncertainties. Variation in the MARR and its effect on the optimal policy are vital pieces of information to decision-makers. These may be better modeled as a fuzzy variable or fuzzy number.

Inflation. A number of engineering economic studies discuss the incorporation of inflation and inflation rate in their models. It is tacitly assumed or conceded that the measurement of this variable is precise. This, however, is not the case. We know that there is a considerable degree of uncertainty due to the way that it is measured. For example, the Consumer Price Index presently used by the government of the United States of America is now under review due to the concern expressed by certain economists. It is argued that the "basket" of goods and services it uses may not accurately reflect the *true* inflation (see, for example, [Samuelson, 1996]). The lack of specificity or precision involved in the measurement of the inflation rate may necessitate the injection of fuzzy modeling such as the use of fuzzy numbers to represent it.

13.4 ECONOMIC LIFE OF AN ASSET MODEL

In some replacement decisions, an asset is required for a long period of time. In these cases, an infinite horizon can be assumed and the decision variable becomes the life of the asset. This is commonly called the *economic life of an asset*. The chosen replacement cycle is the cycle corresponding to the minimum annual equivalent (AE) cost of owning and operating the asset [Sharp-Bette and Park, 1990]. An infinite sequence of replacements and stationary cash flows (with respect to installation time) is assumed.

The general deterministic n-period replacement cycle gives the following AE cost:

$$AE_n(i) = (P - S_n)(A/P, i, n) + S_n i + (A/P, i, n) \sum_{n'=1}^{n} (C_{n'}(P/F, i, n')) \quad (13.3)$$

where

$$\begin{array}{rcl} i & \equiv & \text{minimum attractive rate of return (MARR)} \\ P & \equiv & \text{initial purchase price} \\ S_n & \equiv & \text{salvage value at end of period } n \\ C_{n'} & \equiv & \text{aggregate cash outflow at end of period } n' \\ (A/P, i, n) & \equiv & \text{capital recovery factor} \\ (P/F, i, n) & \equiv & \text{present worth factor} \end{array}$$

Two significant factors determining the optimal replacement cycle are the *aggregate cash flows* at each time period and the MARR. The MARR is set by the organization and is considered a crisp (deterministic) number in the model. The future aggregate cash flows and salvage values, however, are a source of considerable uncertainty and are modeled as triangular fuzzy numbers. These parameters are represented as fuzzy versions of their original counterparts by $\tilde{C}_{n'}$ and $\tilde{S}_n$, respectively. The decision-maker determines a best, worst, and most likely estimate for each. This method of elicitation is quick and has been used previously in replacement analysis [Lohmann, 1986]. These estimates (l, m, and r, respectively) are transformed into a TFN defined as $[l, m, r]$.

Generalizing to consider fuzzy cash flows and salvage values, one obtains the Possibilistic Economic Life of an Asset Model (PELAM) [Hearnes, 1995]. The fuzzy economic life of an asset is defined as the replacement cycle, n, corresponding to the minimum *fuzzy AE* (FAE) of all possible replacement cycles. The traditional model in Equation 13.3 is manipulated into a *proper representation*—i.e., all fuzzy numbers appear only once in the equation:

$$FAE_n(i) = P(A/P, i, n) + \tilde{S}_n\,(i - (A/P, i, n)) + (A/P, i, n) \sum_{n'=1}^{n} \left(\tilde{C}_{n'}(P/F, i, n')\right) \quad (13.4)$$

where

$$\begin{array}{rcl} \tilde{S}_n & \equiv & \text{TFN representing the ``salvage value at end of period } n\text{''} \\ \tilde{C}_{n'} & \equiv & \text{TFN representing the ``aggregate cash flow at end of period } n'\text{''} \end{array}$$

The operations used in PELAM are scalar multiplication, addition, and subtraction. Therefore, the use of TFNs to model the cash flows gives TFNs as a result. However, if the MARR is also modeled as a fuzzy number, the result is not a TFN and the linear approximation to TFN multiplication and the vertex method must be used.

An Example Problem

> An asset to perform service A is required by XYZ corporation indefinitely. Asset B can be purchased for \$50 (all dollar amounts are in thousands) and has a physical life of 5 years. The aggregate cash flows (operating costs - operating revenues) for each year of the life of asset B are \$3, \$4, \$6, \$10, and \$12, respectively, for $n = 1, \ldots, 5$. If the asset is sold at the end of the year, its salvage value is \$35, \$30, \$27, \$23, and \$20, respectively, for $n = 1, \ldots, 5$. Assume a MARR of 10%. Determine the economic life of Asset B.

The assumption of certainty in future cash flows is unrealistic, except in some cases such as when the asset is covered by a service contract. Likewise, future salvage values are dependent on the state of the equipment at that time, possible technological breakthroughs that have occurred, and numerous other uncertain events. The deterministic data is treated as the "most likely" estimates. The local expert or decision-maker provides additional information, namely the "best" and "worst" estimates:

> Being pessimistic, the decision-maker believes that the cash flows (operating costs - revenues) might be much higher than the "most likely" estimates, and the salvage values might be much lower. Therefore the high estimates for the cash flows are \$5, \$7, \$10, \$15, and \$18, respectively, for $n = 1, \ldots, 5$. The low estimates remain near the "most likely" estimates—\$2, \$3, \$5, \$8, and \$10. The salvage values high estimates are \$38, \$32, \$29, \$27, and \$25. The low estimates are \$32, \$24, \$21, \$18, and \$15.

Table 13.1 Results of PELAM for Example Problem.

Replacement Cycle (Years)	Fuzzy AE Cost of Replacement Cycle
1	[19.00, 23.00, 28.00]
2	[16.05, 18.00, 23.33]
3	[14.58, 16.19, 20.94]
4	[14.22, 16.30, 20.75]
5	[14.30, 16.46, 21.09]

Table 13.1 gives the FAE costs for each replacement cycle, $n = 1, \ldots, 5$. Of these, the "minimum" must be chosen. Comparing alternatives described by fuzzy numbers requires a ranking method. All ranking methods reported in the literature suffer from some pathological examples where the result is counterintuitive [Bortolan and Degani, 1985; Chiu and Park, 1994]. The rankings

Table 13.2 Ranking of FAE Costs in Example Problem by Various Methods.

Ranking Method	Ranking
Adamo, $\alpha = 0.9$	$FAE_3 \sim FAE_4 \sim FAE_5 < FAE_2 < FAE_1$
Chang	$FAE_3 < FAE_4 < FAE_5 < FAE_2 < FAE_1$
Chiu and Park, $w = 0.3$	$FAE_4 < FAE_3 < FAE_5 < FAE_2 < FAE_1$
Choobineh and Behrens	$FAE_4 < FAE_3 < FAE_5 < FAE_2 < FAE_1$
Kaufmann and Gupta	$FAE_4 < FAE_3 < FAE_5 < FAE_2 < FAE_1$
Traditional model	$AE_3 < AE_4 < AE_5 < AE_2 < AE_1$

of selected methods are given in Table 13.2. Note that the rankings are not all in agreement as shown in Table 13.2. However, several of them do agree with each other (Chiu and Park, Choobineh and Behrens, and Kaufmann and Gupta). All three of these methods use a ranking function based on the l, m, and r values of the TFN. In the remainder of this paper, the Kaufmann and Gupta method is used as the preferred ranking function.

The Kaufmann and Gupta method is a hierarchical test. It can be described as follows: Let $M_1 = [l_1, m_1, r_1]$ and $M_2 = [l_2, m_2, r_2]$ be two different TFNs.

TEST 1: Compare the *ordinary numbers*: If $0.25(l_1 + 2m_1 + r_1) < 0.25(l_2 + 2m_2 + r_2)$ then $M_1 < M_2$ else if $0.25(l_1 + 2m_1 + r_1) > 0.25(l_2 + 2m_2 + r_2)$ then $M_1 > M_2$, else go to TEST 2.

TEST 2: Compare the *modal values*: If $m_1 < m_2$ then $M_1 < M_2$ else if $m_1 > m_2$ then $M_1 > M_2$ else go to TEST 3.

TEST 3: Compare the *divergence*: If $(r_1 - l_1) < (r_2 - l_2)$ then $M_1 < M_2$ else $M_1 > M_2$.

This method is usually used because (1) it is relatively easy to compute and (2) it always chooses a maximum when the two TFNs are not equal. The latter property is especially important in models based on dynamic programming where a unique optimal value at each stage is desired.

The Optimal Replacement Cycle for the Example Problem

We now show the determination of the optimal replacement policy for the example problem. Using the Kaufmann and Gupta ranking method for reasons discussed above, the optimal replacement cycle for Asset B is 4 years with a fuzzy annual equivalent cost of $[14.22, 16.30, 20.75]$. The modal values of the TFNs for each replacement cycle correspond to the *deterministic AE* costs from

the traditional economic life of an asset model. Therefore, the traditional optimal replacement cycle is immediately available from the fuzzy solution, and the fuzzy solution using fuzzy numbers is equivalent to performing sensitivity analysis on all uncertain variables. It is also interesting to note that using the traditional model the optimal replacement cycle is 3 years, which is different from the 4 years determined by PELAM. Additionally, PELAM determines the optimal replacement cycle based on the decision-maker's estimates of the uncertainty and therefore provides a more informative answer than the traditional model.

13.5 DYNAMIC PROGRAMMING FORMULATION OF ECONOMIC LIFE OF AN ASSET

Bellman formulated a discounted DP approach to determining the optimal replacement cycle of an asset [Bellman and Dreyfus, 1962]. This approach allows for the modeling of *technological change*, unlike the standard *economic life of an asset* model. Technological change implies that the operating revenues and costs, as well as purchase and salvage values, change over time due to technological improvements. An excellent example is the personal computer. As technology improves it affects the purchase price and salvage values of existing computers. The revenues and costs may also be affected by the faster processors.

In the traditional model, at each time period $n = 0, 1, 2, \ldots$, the choice is made to either keep the existing asset or purchase a new one. The following model is an adaptation of Bellman's model [Bellman and Dreyfus, 1962] to the variables and data used in the economic life of an asset model in Section 13.4. This model also uses the year-end convention for operating cash flows. The functional equation is defined:

$f(t) \equiv$ total discounted return (TDR) over an infinite horizon starting with a machine of age t and using an optimal policy.

Thus

$$f(t) = \max \begin{bmatrix} PURCHASE: & S_t - P + \alpha\,(f(1) - C_1) \\ KEEP: & \alpha\,(f(t+1) - C_{t+1}) \end{bmatrix} \tag{13.5}$$

where

$\alpha \equiv$ discounting rate $= (1 + MARR)^{-1}$
$P \equiv$ initial purchase price
$S_n \equiv$ salvage value at the end of period n
$C_n \equiv$ aggregate cash outflow at end of period n

Noting that an optimal policy has the form of keeping a new machine until it is T years old and then replace it with a new machine results in an analytical

solution of the optimal life cycle

$$T = \text{argmax}_{N=1,2,\ldots} \left\{ \begin{array}{ll} \frac{(S_1 - P - C_1)}{1-\alpha} & \text{if } N = 1 \\ \frac{\alpha^{N-1}(S_N - P) - \alpha^N C_1 - \sum_{n=1}^{N-1} \alpha^{n-1} C_{n+1}}{1-\alpha^N} & \text{otherwise} \end{array} \right\} \tag{13.6}$$

with the optimal total discounted return of a new machine being

$$f(0) = -C_1 + \alpha \left[\max_{N=1,2,\ldots} \left\{ \begin{array}{ll} \frac{(S_1 - P - C_1)}{1-\alpha} & \text{if } N = 1 \\ \frac{\alpha^{N-1}(S_N - P) - \alpha^N C_1 - \sum_{n=1}^{N-1} \alpha^{n-1} C_{n+1}}{1-\alpha^N} & \text{otherwise} \end{array} \right\} \right] \tag{13.7}$$

Generalizing this model, the discount factor α is related to the MARR and remains a crisp number. The future aggregate cash flows and salvage values, however, are a source of considerable uncertainty. These are modeled as TFNs and represented as fuzzy versions of their original counterparts by the tilde. Generalizing to consider fuzzy cash flows and salvage values gives the following:

$$T = \text{argmax}_{N=1,2,\ldots} \left\{ \begin{array}{ll} \frac{\left(\tilde{S}_1 - P - \tilde{C}_1\right)}{1-\alpha} & \text{if } N = 1 \\ \frac{\alpha^{N-1}\left(\tilde{S}_N - P\right) - \alpha^N \tilde{C}_1 - \sum_{n=1}^{N-1} \alpha^{n-1} \tilde{C}_{n+1}}{1-\alpha^N} & \text{otherwise} \end{array} \right\} \tag{13.8}$$

with the optimal total discounted return of a new machine being

$$f(0) = -\tilde{C}_1 + \alpha \left[\max_{N=1,2,\ldots} \left\{ \begin{array}{ll} \frac{\left(\tilde{S}_1 - P - \tilde{C}_1\right)}{1-\alpha} & \text{if } N = 1 \\ \frac{\alpha^{N-1}\left(\tilde{S}_N - P\right) - \alpha^N \tilde{C}_1 - \sum_{n=1}^{N-1} \alpha^{n-1} \tilde{C}_{n+1}}{1-\alpha^N} & \text{otherwise} \end{array} \right\} \right] \tag{13.9}$$

An Example Problem

Using the same data as in the example problem in Section 13.4 above, the total discounted returns for the DP formulation are given in Table 13.3 below.

The Optimal Replacement Cycle for the Example Problem

In this example the optimal replacement cycle is 3 years for the fuzzy version under all five ranking methods. By using fuzzy numbers that have a modal value equivalent to the deterministic value of each variable, the solution to the traditional formulation of Bellman's model are the modal values of the respective fuzzy TDRs. Thus the traditional model also has an optimal replacement cycle of 3 years.

Table 13.3 Results of Possibilistic Version of Bellman's Model for the Example Problem.

Replacement Cycle (Years)	Fuzzy TDR of Replacement Cycle
1	$[-230.00, -180.00, -140.00]$
2	$[-186.67, -131.91, -111.90]$
3	$[-164.71, -115.02, -98.34]$
4	$[-164.99, -117.60, -95.90]$
5	$[-170.01, -120.45, -97.71]$

13.6 THE GENERAL SINGLE ASSET REPLACEMENT PROBLEM

When a service is required for only a finite period or the aggregate cash flows are nonstationary with respect to installation time, a more general approach than PELAM is needed. The general single asset replacement problem is widely studied in the literature [Bean et al., 1985; Dreyfus and Law, 1977; Lohmann, 1986; Oakford et al., 1984]. It may be defined as follows:

$$\begin{aligned} a_n &\equiv \text{asset in use at time period } n \\ N &\equiv \text{number of time periods that service is required} \\ A_n &\equiv \text{number of challenging assets at time period } n \end{aligned}$$

The time periods $n = 0, 1, \ldots, N$ represent the periodic replacement decisions. If N is finite, the problem is a finite horizon replacement problem. Otherwise, the problem is an infinite horizon problem. The existing asset is known as the *defender* and can only be placed into service at period 0. The A_0 assets available for replacement at time 0 are known as *current challengers*, and the A_n assets available at future periods are known as *future challengers*. For each period n in the lifetime of each asset, there are three component cash flows describing the *installation cost and/or salvage value*, *operating revenues*, and *operating costs* at period n. The component cash flows of the future challengers are related to the corresponding component cash flow of a current challenger by a scalar function $f(a, n, C)$ where n is the time period in which asset a is installed and $C \in \{1, 2, 3\}$ represents the respective component cash flow. This function allows for the modeling of inflation, technological improvements, and other time-dependent effects on cash flows. For example, to model a crisp 3% inflation rate, then $f(a, n, C) \equiv 1.03^n$ for all a, C. Other more complicated variations can be defined to model a wide range of factors. The component cash flows and relation functions are either known with certainty or estimated by the decision-maker and may also be a fuzzy variable. The problem is to find the sequence(s) of keep/replace decisions that maximize net present value (NPV).

The Possibilistic Model for Single Asset Replacement via dynamic programming (PMSAR) [Hearnes, 1995] is a generalization of SREDM in [Oakford et al., 1984] to allow for fuzzy parameters such as aggregate cash flows, inflation, or technological change. SREDM used the "best", "worst", and "most likely" estimates of the parameters, as in PERT analysis, to create triangular probability distributions. Using those probability distributions, Monte-Carlo simulation provided estimates of the probability of each asset being the optimal choice at period 0. Under such conditions of estimating the distributions, it is arguable that a possibility theory approach is more appropriate. Like PELAM, PMSAR uses TFNs for cash flows. However, PMSAR does not assume an infinite horizon and allows for replacement by competing assets. There is also the possibility of having uncertainty in technological improvements, inflation, or other aspects of the future challengers, which is also modeled as a TFN through a relation function f. The solution technique is a forward dynamic program that uses the Kaufmann and Gupta ranking method to determine the optimal decision and functional equation value at each time period.

We now present a fuzzy analog of the SREDM model. For ease of exposition, we define the following variables of the model:

1. Let k be the state of the system, $k = 1, 2, \ldots, N$, which represents the number of periods of required service.

2. There are two decision variables: (1) n, the period to place an asset into service and (2) a, the asset to place into service.

3. The immediate reward $r(a, n, k)$ is the FPV generated by placing asset a into service at period n and keeping it in service until period k. Defining L_a as the physical lifetime, in time periods, of asset a gives:

$$r(a,n,k) = \begin{cases} \sum_{C=1}^{3} FPV(a,n,C,i,k-n) & \text{if } k-n \leq L_a \\ -M & \text{otherwise} \end{cases} \tag{13.10}$$

   where $M >> 0$ is some sufficiently large number and

   $FPV(a, n, C, i, k-n)$ $\equiv$ Fuzzy Present Value of the *installation cost and/or salvage value*, *operating revenues*, and *operating costs* for $C = 1, 2, 3$, respectively, of placing asset a into service at period n for the remaining $k - n$ time periods using an MARR of i.

4. The transition function $\tau(a, n, k)$ for placing asset a into service at time n for the remaining $k - n$ time periods is $\tau(a, n, k) = n$.

Define the FPV of the functional for this process as the value obtained using an optimal replacement policy from state 0 to state k. Invoking Bellman's Principle of Optimality results in the following functional equation of dynamic programming:

$$FPV^{\star}(k) = \max_{n,a} \{r(a,n,k) + FPV^{\star}(\tau(a,n,k))\} \tag{13.11}$$

where

$$n \in \{0,1,\ldots,k-1\} \text{ and } a \in \begin{cases} \{0,1,\ldots,A_0\} & \text{if } n = 0 \\ \{1,2,\ldots,A_n\} & \text{if } n > 0 \end{cases}$$

and boundary condition $FPV^{\star}(0) = [0,0,0]$. The max operation is performed through the Kaufmann and Gupta ranking method on TFNs.

Relating the parameters of future challengers to the parameters of current challengers via a fuzzy relation function $f(a,n,C)$ is a desirable feature since there may exist considerable uncertainty of the nonprobabilistic nature in future events. This addition is not without its price, however. The model requires multiplication of two fuzzy numbers, which is only weakly distributive and is not closed over TFNs. This problem may be readily circumvented by the adroit use of the vertex method [Dong and Shah, 1987] and a TFN approximation to the product of two TFNs [Kaufmann and Gupta, 1988].

An Example Problem

Let us now consider an adaptation of a replacement problem discussed by Lohmann [Oakford et al., 1984] in which we specifically incorporate fuzzy uncertainty and use it as a vehicle for illustrating our point.

> Three current challengers, $A_n = 1,2,3$ for $n = 1,\ldots,15$, can replace the defender, $a = 0$. The time horizon of $N = 15$ years is established. Each challenger has a physical lifetime of 5 years. The defender has a remaining life of 3 years. For capital transfers, the cash flow at period 0 is the purchase cost and the cash flows at periods $n > 0$ are salvage values. The component cash flows for the most likely estimates plus or minus a percentage are listed in Table 13.4. Assume a MARR of 10%. Suppose the tax rate is assumed to be 50% and MACRS depreciation tables for a 7-year recovery period are utilized in the determination of depreciation tax shield.

The optimal sequence of decisions is determined via forward dynamic programming with rewards modeled as fuzzy numbers. The functional equation

Table 13.4 Data for Example Problem.

a	n	$C = 1$	$C = 2$	$C = 3$
0	0	5423	0	0
	1	4194 ±5%	-15791 ±5%	19200 ±5%
	2	3355 ±5%	-17685 ±5%	19200 ±5%
	3	2684 ±10%	-19808 ±10%	19200 ±10%
1	0	20000	0	0
	1	16100 ±5%	-8000 ±5%	19200 ±5%
	2	13200 ±5%	-8960 ±5%	19200 ±5%
	3	10840 ±10%	-10035 ±10%	19200 ±10%
	4	8192 ±10%	-11239 ±10%	19200 ±10%
	5	6554 ±10%	-12588 ±10%	19200 ±10%
2	0	21000	0	0
	1	16800 ±5%	-7500 ±5%	19200 ±5%
	2	13440 ±5%	-8400 ±5%	19200 ±5%
	3	10252 ±10%	-9408 ±10%	19200 ±10%
	4	8602 ±10%	-10537 ±10%	19200 ±10%
	5	6881 ±10%	-11801 ±10%	19200 ±10%
3	0	22000	0	0
	1	17600 ±25%	-7250 ±25%	19200 ±5%
	2	14080 ±25%	-8120 ±25%	19200 ±5%
	3	11264 ±25%	-9094 ±25%	19200 ±10%
	4	9011 ±25%	-10186 ±25%	19200 ±10%
	5	7209 ±25%	-11408 ±25%	19200 ±10%

for the first decision stage is computed as follows:

$$
\begin{aligned}
FPV(0,0,1,0.1,1) &= \text{- Purchase + PV(Salvage)} \\
&\quad \text{+ PV(Tax shield from depreciation)} \\
&\quad \text{- PV(Tax on gain at sale)} \\
&= -[5423,5423,5423] + [3984.3,4194,4403.7] \cdot (1.1)^{-1} \\
&\quad +[5423,5423,5423] \cdot 0.5 \cdot 0.1429 \\
&\quad -([3984.3,4194,4403.7] \\
&\quad -[5423,5423,5423] \cdot (1-0.1429)) \\
&= [-1146.95,-1051.64,-956.32] \\
FPV(0,0,2,0.1,1) &= \text{PV(After-tax costs)} \\
&= [-7536.61,-7177.73,-6818.84] \\
FPV(0,0,3,0.1,1) &= \text{PV(After-tax revenues)} \\
&= [8290.91,8727.27,9163.64]
\end{aligned}
$$

Note that the salvage and purchase values and functional relation (if used) appear more than once in the FPV calculations for $C = 1$, therefore the vertex method is used to generate the correct FPV. This entails computing deterministically the problem for the modal (most likely values) as well as an additional 2^3 times for all the possible combinations of "best" and "worst" estimates for these parameters. This gives the following for $N = 1$:

$$\begin{aligned} FPV^{\star}(1) &= \max_{n=0,a\in\{0,1,2,3\}} \left\{ \sum_{C=1}^{3} FPV(a,0,C,0.1,1) + FPV^{\star}(0) \right\} \\ &= \max \left\{ \begin{array}{ll} [-392.66, 497.91, 1388.48] + [0,0,0] & \text{for } a=0, n=0 \\ [515.91, 1500.00, 2484.09] + [0,0,0] & \text{for } a=1, n=0 \\ [511.36, 1500, 2488.64] + [0,0,0] & \text{for } a=2, n=0 \\ [-228.41, 1431.82, 3092.04] + [0,0,0] & \text{for } a=3, n=0 \end{array} \right\} \\ &= [515.91, 1500.00, 2484.09] \text{ for } a=1. \end{aligned} \tag{13.12}$$

The same calculations are performed for the remaining decision stages, up to $k = 15$, recording both the functional equation value and the optimal decision for each stage as in Table 13.5.

Table 13.5 Dynamic Programming Results for Basic Example Problem.

N	Optimal Policy (asset, time installed)	$FPV^{\star}(N)$
1	(1,0)	[515.91,1500.00,2484.09]
2	(1,0)	[1686.70,3159.42,4632.15]
3	(1,0)	[2054.50,4759.95,7465.39]
4	(2,0)	[2799.62,6360.66,9921.71]
5	(3,0)	[413.48,7715.23,15017.00]
6	(3,0) (2,5)	[731.00,8646.61,16562.20]
7	(3,0) (1,5)	[1460.79,9676.98,17893.20]
8	(2,0) (2,4)	[4711.79,10705.10,16698.40]
9	(3,0) (2,5)	[2151.82,11664.70,21177.60]
10	(3,0) (3,5)	[670.22,12505.80,24341.40]
11	(3,0) (3,5) (2,10)	[867.37,13084.10,25300.80]
12	(3,0) (3,5) (1,10)	[1320.51,13723.90,26127.20]
13	(3,0) (2,5) (2,9)	[3339.13,14362.20,25385.40]
14	(3,0) (3,5) (2,10)	[1749.59,14958.10,28166.60]
15	(3,0) (3,5) (3,10)	[829.63,15480.30,30131.10]

The Optimal Replacement Policy for the Example Problem

As earlier we show the determination of the optimal replacement problem for the model example under fuzziness. Solving the functional equation given in Equation 13.11 for $N = 15$ results in the optimal replacement policy:

$$(3,0)(3,5)(3,10) \text{ with } FPV^{\star}(15) = [829.63, 15480.30, 30131.10].$$

This translates into buying asset 3 at time 0, again at time 5, and again at time 10. In this solution, the modal values of each $FPV^{\star}(k)$ represent the $NPV^{\star}(k)$ of the deterministic model, while the lower and upper values indicate the overall uncertainty in the decision. The uncertainty signified by the width of the base of each $FPV^{\star}(k)$ also is equivalent to the range of possible values determined via multivariable sensitivity analysis. Thus a fuzzy model immediately provides both the traditional deterministic NPV for this policy as well as the range of values the NPV may take due to uncertainty in the parameters.

Other uncertain factors may be introduced via the relation function $f(a, n, C)$. If, for example, a moderate inflation increase of $[1\%, 2\%, 3\%]$ per year were expected while the other parameters remained the same, the optimal replacement policy becomes:

$$(3,0)(2,5)(2,10) \text{ with } FPV^{\star}(15) = [-969.74, 16558.10, 34313.60].$$

Contrast this with the traditional stochastic model, SREDM, where the solution is derived using Monte-Carlo simulation. A large number of realizations of the uncertain (and assumed random) variables are generated and each set of them is solved deterministically [Oakford et al., 1984]. From this large sample, the probability that each alternative current asset is the optimal first choice can be estimated (see Figure 13.1) as well as the corresponding cumulative probability distributions of (1) the economic life of each current asset, (2) the NPV of the optimal sequence of challengers for a finite horizon (see Figure 13.2, and (3) the equivalent finite horizon time for infinite horizon problems. We note that the probability distributions generated by SREDM are *subjectively* interpreted to determine the optimal *current* decision and no information regarding future decisions is available.

13.7 INSPECTION AND REPLACEMENT MODELS

Another problem of great interest and relevance is inspection and replacement under uncertainty. In real-life problems, it is more common to have a combination of inspection and replacement, with elements of uncertainty in each. Dreyfus and Law [Dreyfus and Law, 1977] treat this situation stochastically. There do not exist any analogous models involving either a possibilistic or fuzzy

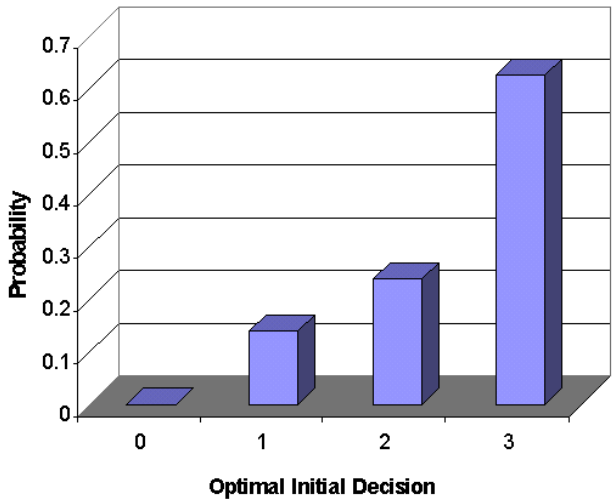

Figure 13.1 Monte-Carlo simulation results for the example problem depicting the probability distribution for the optimal first choice.

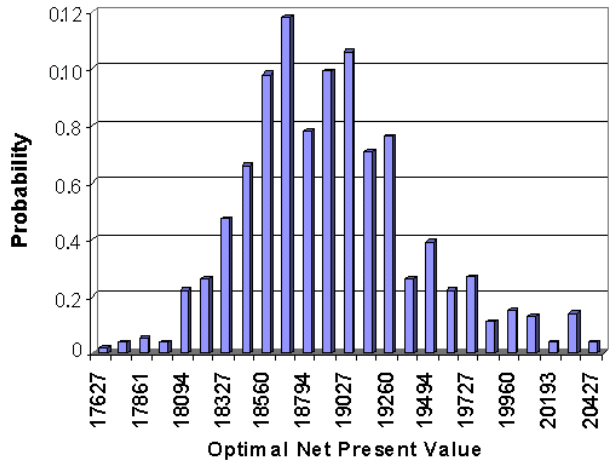

Figure 13.2 Monte-Carlo simulation results for the example problem depicting the probability distribution for the optimal NPV.

approach to this problem. For ease of presentation, let us examine the stochastic model using it as the leitmotif for discussing the role of fuzziness in this more realistic situation.

In the inspection and replacement model, parts are inspected at the beginning of a time period and fail at the end of a time period. Parts that are inspected and judged to be in working order are assumed to work just as well as a new part and are given an age of 0. Because of the series configuration, any part failing in the system causes the entire system to fail. The probability that a part j is good at the beginning of a period given that it was also good at the beginning of the previous period is p_j. For inspection of a single part j it takes c_j time periods. Inspection of multiple parts $j_1, \ldots, j_m$ saves time and takes $c_{j_1 \ldots j_m} < c_{j_1} + \cdots + c_{j_m}$. Similarly, replacement takes r_j and $r_{j_1 \ldots j_m} < r_{j_1} + \cdots + r_{j_m}$ time periods. Let the parts have initial ages $t_0(j_1), \ldots, t_0(j_m)$. The objective is to maximize the expected number of periods that the system remains in working order in the next N periods.

As an example, let $m = 2$. Several DP functional equations are needed to clarify the model:

$f_n(g_1, g_2)$ = Maximum expected number of periods from period n through N that the system is working, given that period n starts with parts 1 and 2 last known to be in good order g_1 and g_2 periods previously.

For convenience let

$K_n(g_1)$ = Maximum expected number of periods from period n through N that the system is working, given that period n starts with parts 1 last known to be in good order g_1 and part 2 has just been inspected and found to be in a failed condition

$L_n(g_2)$ = Maximum expected number of periods from period n through N that the system is working, given that period n starts with parts 2 last known to be in good order g_2 and part 1 has just been inspected and found to be in a failed condition

M_n = Maximum expected number of periods from period n through N that the system is working, given that period n starts with parts 1 and 2 just inspected and found to be in a failed condition

The possible actions are to leave the system alone (LA), inspect any combination of parts (I_1, I_2, I_{12}), or replace any combination of parts (R_1, R_2, R_{12}).

This gives

$$f_n(g_1,g_2) = \max \begin{bmatrix} LA: & p_1^{g_1}p_2^{g_2} + f_{n+1}(g_1+1, g_2+1) \\ I_1: & p_1^{g_1} f_{n+c_1}(0,g_2) + (1-p_1^{g_1})L_{n+c_1}(g_2) \\ I_2: & p_2^{g_2} f_{n+c_2}(g_1,0) + (1-p_2^{g_2})K_{n+c_2}(g_1) \\ I_{12}: & p_1^{g_1}p_2^{g_2} f_{n+c_{12}}(0,0) + (1-p_1^{g_1})p_2^{g_2} L_{n+c_{12}}(g_2) \\ & +p_1^{g_1}(1-p_2^{g_2})K_{n+c_{12}}(g_1) + (1-p_1^{g_1})(1-p_2^{g_2})M_{n+c_{12}} \\ R_1: & f_{n+r_1}(0,g_2) \\ R_2: & f_{n+r_2}(g_1,0) \\ R_{12}: & f_{n+r_{12}}(0,0) \end{bmatrix}$$

$$K_n(g_1) = \max \begin{bmatrix} I_1: & p_1^{g_1} K_{n+c_1}(0) + (1-p_1^{g_1}) f_{n+c_1+r_{12}}(0,0) \\ R_2: & f_{n+r_2}(g_1,0) \\ R_{12}: & f_{n+r_{12}}(0,0) \end{bmatrix}$$

$$L_n(g_2) = \max \begin{bmatrix} I_1: & p_2^{g_2} L_{n+c_2}(0) + (1-p_2^{g_2}) f_{n+c_2+r_{12}}(0,0) \\ R_2: & f_{n+r_1}(0,g_2) \\ R_{12}: & f_{n+r_{12}}(0,0) \end{bmatrix}$$

and

$$M_n = f_{n+r_{12}}(0,0)$$

with boundary conditions for $n > N$ given as

$$f_n(g_1,g_2) = K_n(g_1) = L_n(g_2) = M_n = 0 \quad \forall g_1, g_2$$

There are several questions that can be raised regarding the assumptions of this model. First, the inspections are assumed to be perfect in that no misdiagnoses are made. This is rarely the case in the real world. Varying degrees of state imperfections can be represented via fuzzy modeling. Second, the "working" state is rarely as good as the original new part and therefore the inspection should tell to what degree the machine is working. This again can be more realistically modeled as a fuzzy variable with the membership function being a surrogate for the extent to which the machine is in a "working" order. Finally, the probability distributions used may not be known or of a familiar type and therefore may need to be replaced by possibility distributions. This model also suffers from the "curse of dimensionality" in that the state and action spaces grow exponentially with the number of parts m in the system.

13.8 MAINTENANCE DECISIONS

Not all replacement decisions are limited to simply choosing to "keep" or "replace" the existing asset. Fortunately, the DP formulation is sufficiently general to allow for other options. For example, a third option may be the "purchase

of a used machine". Bellman describes the DP formulation of the replacement model with the option to purchase a used machine as well as that of a model with an "overhaul" option [Bellman and Dreyfus, 1962]. The overhaul option can be done either in a general manner, where the cash flows and other variables are functions of both the installation time and the overhaul time, or by allowing the overhauled asset (t years old) to maintain the characteristics of a younger asset ($t' < t$ years old).

Bellman's general model of overhauling machines is formulated in [Bellman and Dreyfus, 1962]. We briefly describe it below. To model overhauls, another dimension is added to the problem. Let the functional equation be:

$$\begin{aligned} f_N(t_1, t_2) \quad = \quad & \text{value at year } N \text{ of the overall discounted return from a} \\ & \text{machine of age } t_1, \text{ last overhauled at age } t_2, \text{ where an} \\ & \text{optimal replacement policy is employed for the} \\ & \text{remainder of the process.} \end{aligned} \tag{13.13}$$

Then the recurrence relation becomes:

$$f_N(t_1, t_2) = \max \begin{bmatrix} R: & -C_N(0,0) + S_N(t_1, t_2) - P + \alpha f_{N+1}(1, 0) \\ K: & -C_N(t_1, t_2) + \alpha f_{N+1}(t_1 + 1, t_2) \\ O: & -C_N(t_1, t_2) - O_N(t_1, t_2) + \alpha f_{N+1}(t_1 + 1, t_1) \end{bmatrix} \tag{13.14}$$

where $O_N(t_1, t_2)$ is the cost of overhauling in year N a machine of age t_1 last overhauled at age t_2. Using fuzzy numbers to model the aggregate cash flows, overhaul costs, and salvage values increases the number of calculations by a factor of 3. Thus the general complexity of the fuzzy version remains unchanged from the traditional model.

Modeling an overhaul option in PMSAR can be done either by defining functions that relate an overhauled asset's aggregate cash flows and salvage value to its installation time and overhaul time or defining challenging assets that represent the costs and characteristics of an overhauled machine. By defining a function, an extra dimension is added to the model as above, while defining "overhaul" challengers keeps the solution technique the same as in the original PMSAR.

We return to the example problem for the PMSAR above to illustrate a maintenance option with an imprecise or vague effect. Suppose that the maintenance option is modeled as a "ghost" asset which is defined as follows: The purchase cost P' of the "ghost" asset a' is a function of the cost of the maintenance option on the asset a currently in place and the number of years n' since a was installed.

$$P' = ([0.15, 0.20, 0.25] \cdot P_a)(1.03)^{n'}$$

where P_a is the purchase price of asset a. This particular function states that the base maintenance cost is a fuzzy number that is $[15\%, 20\%, 25\%]$ of the original cost of the asset with an increase of 3% per year since installation. The result of the maintenance is also fuzzy, a $[6\%, 10\%, 12\%]$ reduction in the original operating costs for the following year. The component cash flows corresponding to this maintenance challenger a' are a function of the time n' that the maintenance occurs as well as the original asset and its installation time. Let us denote performing maintenance on the existing asset at period n as (M, n). The resulting optimal maintenance and replacement policy for $N = 15$ then becomes:

$$(M,0)(M,1)$$
$$(1,2)(M,3)(M,4)(M,5)(M,6)$$
$$(1,7)(M,8)(M,9)(M,10)(M,11)$$
$$(2,12)(M,13)(M,14)$$

with an optimal fuzzy present value $FPV^{\star}(15) = [25709.00, 40749.90, 54887.80]$. This translates to performing maintenance on the *current defender* at periods 0 and 1, purchasing asset 1 at period 2 and performing maintenance on this asset each period until asset 1 is purchased again in time 7. The maintenance at each period continues on asset 1 until asset 2 is purchased at period 12. Maintenance on asset 2 then performed at each remaining period. The fuzzy present value with this maintenance option, in this example, has risen significantly.

13.9 SUMMARY

Probability theory has been used as the traditional approach for modeling uncertainty in economic analysis. This is acceptable only to the extent that uncertainty is satisfactorily equated with randomness. However, there exist other types of uncertainty which are especially relevant to economic decision analysis. Thus, there is a role to be played by nonprobabilistic uncertainty as shown in this effort. Many approaches have been shown to be possible. A brief survey of replacement analysis, focusing on the use of nonprobabilistic uncertainty, is given. The use of triangular fuzzy numbers provides a compromise between computational efficiency and realistic modeling of the uncertainty. Thus, this discussion emphasizes fuzzy numbers. In the extension to the *economic life of an asset* model, the uncertainty in the parameters is explicitly modeled. By only a three-fold increase in the number of computations, the optimal choice based on the decision-maker's best estimates of these parameters is easily obtained. The traditional deterministic models are a special case of this new possibilistic model. In effect, PELAM performs multivariable sensitivity analysis on all the uncertain parameters concurrently and incorporates this uncertainty into the determination of the optimal decision. Bellman's DP model is also extended to

use fuzzy numbers with similar benefits. The benefits for PMSAR are virtually the same, except there is a greater increase in the number of computations due to the vertex method. Contrast this with the large number of repetitions that Monte-Carlo simulation requires, as well as the subjective interpretation of the results, and this disadvantage is not too severe.

When probability distributions are not known, or when a stochastic model is too difficult to solve, fuzzy sets and possibility theory offer an efficient alternative in replacement analysis. There are a number of benefits for modeling the uncertainty in the replacement problem via fuzzy numbers. We outline a few of them:

- The use of fuzzy uncertainty may be more appropriate when modeling systems with human subjectivity. The only existing technique in replacement analysis that modeled general uncertainty in the replacement decision was a Monte-Carlo simulation method [Lohmann, 1986].

- Creating a triangular distribution from the best, worst, and most likely estimates of an expert is more appropriate for possibility theory than probability theory due to the lack of probabilistic information.

- The results of each model can be easily plotted to provide a graphical representation of the effects of the uncertain parameters and can provide this information without a significant increase in the computational complexity. The algorithms are easily implemented on a personal computer.

- The use of TFNs retains the triangular distribution throughout the solution algorithm and conveniently shows the best, worst, and most likely NPVs of each possible decision.

- The use of fuzzy numbers corresponds to performing sensitivity analysis on all uncertain parameters simultaneously, as well as identifying the answer to the deterministic version of the problem which corresponds to the modal value of each fuzzy number.

The models detailed here are limited to extensions of standard economic decision-making techniques. They take a cost approach to determining the optimal replacement policy. Fuzzy sets allow other approaches, such as a fuzzy control approach using IF-THEN rules. For instance, a maintenance or replacement decision may depend on several variables such as *operating revenues*, *operating costs*, *inflation rate*, *technological age* (asset's status as a current technology in the industry), *chronological age*, and numerous others. While no precise mathematical models of these more realistic situations, fuzzy sets defined on these linguistic variables could generate a fuzzy rulebase consisting of rules such as the following two examples:

IF *operating revenues* are **MEDIUM** AND *operating costs* are **HIGH**, THEN *preventive maintenance priority* is **MEDIUM**.

IF *inflation rate* is **HIGH** AND *technological age* is **CUTTING EDGE** AND *chronological age* is **YOUNG**, THEN *replacement priority* is **LOW**.

This approach develops a maintenance and replacement policy based on priorities which differs from the traditional NPV approach. These can be readily solved to yield maintenance and replacement policies that are in many cases intuitive and human-friendly. There are, of course, some limitations such as those induced by fuzzification and defuzzification via standard tools. These can however be ameliorated by recourse to optimal defuzzification strategies as discussed by Esogbue and Song [Esogbue and Song, 1995; Esogbue and Song, 1996; Esogbue et al., 1996]. This method could also utilize some of the advances in membership function generation such as reinforcement learning and neural networks as well as other new techniques in fuzzy rulebase control.

References

Alchian, A. (1952). Economic replacement policy. Technical Report Publication R-224, The RAND Corporation, Santa Monica, CA.

Bean, J., Lohmann, J., and Smith, R. (1985). A dynamic infinite horizon replacement economy decision model. *The Engineering Economist*, 30(2):99–120.

Behrens, A. and Choobineh, F. (1989). Can economic uncertainty always be described by randomness? In *Proceedings of the 1989 International Industrial Engineering Conference*, pages 116–120.

Bellman, R. (1955). Equipment replacement policy. *SIAM Journal Applied Mathematics*, 3:133–136.

Bellman, R. and Dreyfus, S. (1962). *Applied Dynamic Programming*. Princeton University Press, Princeton, NJ.

Bezdek, J. (1994). Fuzziness vs. probability - again (!?). *IEEE Transactions on Fuzzy Systems*, 2(1):1–3.

Bortolan, G. and Degani, R. (1985). A review of some methods for ranking fuzzy subsets. *Fuzzy Sets and Systems*, 15:1–19.

Buckley, J. (1987). The fuzzy mathematics of finance. *Fuzzy Sets and Systems*, 21:257–273.

Buckley, J. (1992). Solving fuzzy equations in economics and finance. *Fuzzy Sets and Systems*, 48:289–296.

Chiu, C. and Park, C. (1994). Fuzzy cash flow analysis using present worth criterion. *The Engineering Economist*, 39(2):113–138.

Choobineh, F. and Behrens, A. (1992). Use of intervals and possibility distributions in economic analysis. *Journal of the Operational Research Society*, 43(9):907–918.

Dong, W. and Shah, H. (1987). Vertex method for computing functions of fuzzy variables. *Fuzzy Sets and Systems*, 14:65–78.

Dreyfus, S. and Law, A. (1977). *The Art and Theory of Dynamic Programming*. Academic Press, New York.

Dubois, D. and Prade, H. (1978). Operations on fuzzy numbers. *International Journal of Systems Science*, 9:612–626.

Dubois, D. and Prade, H. (1987). Fuzzy numbers: An overview. In Bezdek, J., editor, *Analysis of Fuzzy Information*, volume 1, pages 3–39. CRC Press, Boca Raton, FL.

Dubois, D. and Prade, H. (1994). Fuzzy sets–a convenient fiction for modeling vagueness and possibility. *IEEE Transactions on Fuzzy Systems*, 2(1):16–21.

Esogbue, A. and Kacprzyk, J. (1996). Fuzzy dynamic programming: A survey of main developments and applications. *Archives of Control Sciences*, 5(1-2):39–59.

Esogbue, A. and Song, Q. (1995). Optimal defuzzification and applications. In *Proceedings of the International Joint Conference on Information Science, Fourth Annual Conference on Fuzzy Theory & Technology*, pages 182–185, Wrightsville Beach, NC.

Esogbue, A. and Song, Q. (Submitted 1996). Defuzzification filters: Design and applications. *IEEE Transactions on Fuzzy Sets*.

Esogbue, A., Song, Q., and Hearnes, W. (1996). Defuzzification filters and applications to power system stabilization problems. In *Proceedings of the 1996 IEEE International Conference on Systems, Man and Cybernetics Information, Intelligence and Systems*, Beijing, China.

Gluss, B. (1972). *An Elementary Introduction to Dynamic Programming: A State Equation Approach*. Allyn and Bacon, Inc.

Gupta, C. (1993). A note on the transformation of possibilistic information into probabilistic information for investment decisions. *Fuzzy Sets and Systems*, 56:175–182.

Hearnes, W. (1995). Modeling with possibilistic uncertainty in the single asset replacement problem. In *Proceedings of the 2nd Joint Conference on Information Sciences*, pages 552–555, Wrightsville Beach, NC.

Kacprzyk, J. (1978). Decision-making in a fuzzy environment with fuzzy termination time. *Fuzzy Sets and Systems*, 1:169–179.

Kaufmann, A. and Gupta, M. (1988). *Fuzzy Mathematical Models in Engineering and Management Science*. Elsevier Science Publishers, B.V.

Klir, G. (1994). On the alleged superiority of probabilistic representation of uncertainty. *IEEE Transactions on Fuzzy Systems*, 2(1):27–31.

Klir, G. and Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall PTR, Upper River Saddle, NJ.

Kosko, B. (1994). The probability monopoly. *IEEE Transactions on Fuzzy Systems*, 2(1):32–33.

Laviolette, M. and Seaman, J. J. (1994). The efficacy of fuzzy representations of uncertainty. *IEEE Transactions on Fuzzy Systems*, 2(1):4–15.

Li Calzi, M. (1990). Towards a general setting for the fuzzy mathematics of finance. *Fuzzy Sets and Systems*, 35:265–280.

Lohmann, J. (1986). A stochastic replacement economic decision model. *IIE Transactions*, 18:182–194.

Oakford, R. (1970). *Capital Budgeting: A Quantitative Evanluation of Investment Alternatives*. The Ronald Press Co., New York.

Oakford, R., Lohmann, J., and Salazar, A. (1984). A dynamic replacement economy decision model. *IIE Transactions*, 16:65–72.

Oakford, R., Salazaar, A., and DiGiulio, H. (1981). The long term effectiveness of expected net present value maximization in an environment of incomplete and uncertain information. *AIIE Transactions*, 13:265–276.

Samuelson, R. (1996). Imperfect vision: A flawed cpi makes it harder for us to read the past and plan for the future. *Newsweek*, 55.

Savage, L. (1954). *The Foundations of Statistics*. Wiley, New York.

Sharp-Bette, G. and Park, C. (1990). *Advanced Engineering Economics*. John Wiley and Sons, Inc.

Terborgh, G. (1949). *Dynamic Equipment Policy*. Machinery and Applied Products Institute, Washington, DC.

Tversky, A. and Kahneman, D. (1971). Belief in the law of small numbers. *Psychological Bulletin*, 2:105–110.

Tversky, A. and Kahneman, D. (1974). Judgement under uncertainty: Heuristics and biases. *Science*, 185:1124–1131.

Tversky, A. and Kahneman, D. (1981). The framing of decisions and the psychology of choice. *Science*, 211:3–8.

Von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Wagner, H. (1975). *Principles of Operations Research*. Prentice-Hall.

Ward, T. (1985). Discounted fuzzy cash flow analysis. In *Proceedings of the 1985 Fall Industrial Engineering Conference*, pages 476–481.

Zadeh, L. (1978). Fuzzy sets as a basis for a theory of possibility. *IEEE Transactions on Systems, Man, and Cybernetics*, 1:3–28.

Zimmerman, H. (1991). *Fuzzy Set Theory and Its Applications*. Kluwer Academic Press, Boston, MA, second edition.